Comparison of some one sample confidence intervals for estimating the mean of the Weibull distribution

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ABSTRACT: In this article, an attempt has been made to review existing interval estimators for Weibull mean and compare them under the same simulation condition. A comparison of the performance of the CI estimators has been made with respect to simulated average widths and average coverage probabilities. A comparative study revealed that the performances of the estimators differ significantly when the sample sizes are small and shape parameter is small i.e. data are from a highly skewed Weibull distribution. Results are illustrated with a real data set for practitioners. Based on the simulation study, good interval estimators have been recommended for use in practice.

KEY WORDS: Confidence interval, Weibull mean, Comparative study


INTRODUCTION

Weibull distribution is widely used in reliability and survival analysis due to its flexible shape and ability to model a wide range of failure rates. It can be derived theoretically as a form of extreme value distribution, governing the time to occurrence of the “weakest link” of many competing failure processes. Its special case with shape parameter \( b = 2 \) is the Rayleigh distribution which is commonly used for modeling the magnitude of radial error when \( x \) and \( y \) coordinate errors are independent normal variables with zero mean and the same standard deviation, while the case \( b = 1 \) corresponds to the widely used exponential distribution.

Let \( X \) follows a Weibull distribution with scale parameter \( a \) and shape parameter \( b \). The probability density function (pdf) of \( X \) is given by:

\[
f_X(x; a, b) = \begin{cases} \frac{b}{a} \left( \frac{x}{a} \right)^{b-1} \exp \left( - \left( \frac{x}{a} \right)^b \right) & : x > 0, a > 0, b > 0; \\ 0 & : \text{otherwise} \end{cases}
\]

The mean of the Weibull distribution is given by

\[
\mu = a \Gamma \left( 1 + \frac{1}{b} \right)
\]

where \( \Gamma(k) = \int_0^\infty m^{k-1} \exp(-m) dm \) is the Gamma function. The problem of interval estimation for the mean \( \mu \) when both parameters \( a \) and \( b \) are unknown has been less attended in the literature. Colosimo and Ho (1999) obtained CI for \( m \) using asymptotic-normal theory and it is usually called Wald CI. Yang et al. (2007) proposed CI for Weibull mean from type-II censored data based on a Chi-square distributed pivotal quantity given by Lawless (1982), involving modified maximum likelihood estimator (MMLE) of \( b \) in the resulting CI and is referred to as naïve CI. An adjustment is suggested to the Chi-square quantile for obtaining more refined results. Krishnamoorthy et al. (2009) utilized generalized variable (GV) approach for developing confidence limits for Weibull mean \( \mu \) and studied coverages of the corresponding one sided confidence limits and compared with coverages of the Wald confidence limits based on log transformed variable.

In this paper, we reviewed and compared the CIs proposed.
by different researchers at different times and found good intervals for estimating the mean of the Weibull distribution. Since a theoretical comparison is not tractable, a simulation study has been made to compare the performance of the estimators. A comparison was made with respect to both simulated average widths and average coverage probabilities.

The article is organized as follows: Section 2 provides a brief review of the existing methods in the literature for CI estimation of the Weibull mean. Section 3 presents the comparison of the CIs based on the simulated average widths and the average coverage probabilities. Section 4 illustrates the methods using a real data set. Section 5 provides overall conclusions and recommendations.

**MATERIALS AND METHODS**

**Existing methods:**

In this section, we reviewed existing methods for CI estimation of the Weibull mean. For convenience, methods discussed in this section are abbreviated and the abbreviations are given in the parentheses following the name of each method.

**Wald CI based on log transformed variable (WALD-LOG):**

Based on the log-transformed variable, the test statistic

\[ W = \frac{\log(\hat{\mu}) - \log(\mu)}{\sqrt{\text{var}[\log(\hat{\mu})]}} \]

can be used to obtain the CI for the mean of the Weibull \((a, b)\) distribution, where,\n
\[ \hat{\mu} = \hat{a} \Gamma(1 + (1/\hat{b})) \]

\(\hat{a}\) is maximum likelihood estimator (MLE) of \((a, b)\) is MLE of \(b\) and \(\text{var}[\hat{\mu}]\) is estimated using delta method as,

\[ \text{var}[\hat{\mu}] = \text{var}[\hat{b}] \left( \frac{\partial \mu}{\partial a} \right)_{a, \hat{b}}^2 + 2c \text{cov}[\hat{a}, \hat{b}] \]

\[ \left( \frac{\partial \mu}{\partial a} \right)_{a, \hat{b}} \left( \frac{\partial \mu}{\partial b} \right)_{a, \hat{b}} + \text{var}[\hat{a}] \left( \frac{\partial \mu}{\partial a} \right)_{a, \hat{b}}^2 \]

The formulae for var[\hat{a}], var[\hat{b}] and \text{cov}[\hat{a}, \hat{b}] given by Cohen (1965) are used with appropriate modifications to adopt for the form of density function used in this work. Employing the approximation

\[ \text{var}[\log(\hat{\mu})] \approx \text{var}[\hat{\mu}] / \hat{\mu}^2 \]

and asymptotic normality of the MLEs, the resulting CI is given by:

\[ \exp\left[ \log(\hat{\mu}) \pm Z_{1-\alpha/2} \sqrt{\text{var}[\log(\hat{\mu})]} \right] \]

where, \(Z_{\alpha}\) is the \(\alpha^{th}\) quantile of standard normal distribution. Krishnamoorthy et al. (2009) studied coverages of the one sided confidence limits of above CI at 5 per cent level of significance.

**Wald CI proposed by Colosimo and Ho (WALD-CH):**

Colosimo and Ho (1999) proposed the following CI for the mean \(\mu\) of the Weibull \((a, b)\) distribution,

\[ \hat{\mu} \pm Z_{1-\alpha/2} \sqrt{\text{var}[\hat{\mu}]} \]

where, \(\text{var}[\hat{\mu}]\) is obtained using equation (1).

**Naive CI based on modified MLE (NMMLE):**

Yang et al. (2007) considered type II censored life testing experiment and used a pivotal quantity \(T(b) = 2S(b) / a^b\) to obtain CI for \(\mu\) where,

\[ S(b) = \sum_{i=1}^{r} x_i^b + (n-r)x_r^b \]

and \(r=1, 2, ..., n\) is the level of censoring. Based on the fact that \(T(b)\) follows a Chi-square distribution with \(2r\) degrees pf freedom (df) (see Lowless (1982)), for known value of an exact CI for \(\mu\) is given by:

\[ \left[ \Gamma(1+1/b) \frac{2S(b)}{\chi^2_{2r,1-\alpha/2}} \right]^{1/b}, \Gamma(1+1/b) \frac{2S(b)}{\chi^2_{2r,1-\alpha/2}} \]

where, \(\chi^2_{V,\alpha}\) is the \(\alpha^{th}\) quantile of Chi-square distribution with \(V\) df. When \(b\) is unknown, it is replaced by its appropriate estimator \(\hat{b}\) in equation (2). Authors referred to this CI as a naive CI and based on comparative study between MLE and MMLE, suggested using MMLE, \(\tilde{b}\), for \(\hat{b}\), which is a solution to the equation:

\[ r \sum_{i=1}^{r} x_i^b \log(x_i) \left[ \sum_{i=1}^{r} x_i^b \right]^{-1} + r \log(x_i) = 0 \]
where, \[ \sum_{i=1}^{r}x_i = \sum_{i=1}^{r}(n-r)x_r \]. In this case, for obtaining more refined results, authors suggested an adjustment to Chi-squared quantile as:

\[ \chi^2_{\nu,\alpha} = c\chi^2_{\nu,\alpha} - 2r(c-1) \]

where, \( c=1+\tau^2\left(k-p_0\log(-\log(1-\alpha))\right)^2/p_0 \)
is the adjustment factor and \( \chi^2_{\nu,\alpha} = c\chi^2_{\nu,\alpha} - 2r(c-1) \)is \( p_0 \) is the adjustment factor and \( p_0 = r/n \). Here, \( \tau^2 \) and \( k \) are functions of \( p_0 \) and authors have tabulated their values for \( p_0 = 0.1, 0.2, ..., 1 \) based on a simulation study. For uncensored data, \( p_0 = 1 \) for which \( \tau^2 = 0.607927 \) and \( k = 0.422642 \).

**CI based on GV approach (GVA):**

Krishnamoorthy et al. (2009) have used the GV approach developed by Tsui and Weerahandi (1989) and Weerahandi (1993) for construction of CI for Weibull mean. Here, authors obtained the generalized pivotal quantity (GPQ) separately for the parameters \( \alpha \) and \( \beta \) as:

\[ G_{\alpha} = (1/\hat{\alpha}^*)\left(b^*/\hat{b}^*\right)\hat{\alpha}_0 \quad \text{and} \quad G_{\beta} = \hat{b}_0/\hat{b}^* \]

Here \( \hat{\alpha}_0 \) and \( \hat{b}_0 \) are MLEs of \( \alpha \) and \( \beta \) respectively based on the observed sample and \( \hat{\alpha}^* \) and \( \hat{b}^* \) are MLEs based on a random sample of size \( n \) from Weibull (1,1) distribution. Then GPQ for the mean \( \mu = a\Gamma(1+(1/b)) \) is given by:

\[ G_{\mu} = G_{\alpha}\Gamma(1+1/G_b) = (1/\hat{\alpha}^*)\left(b^*/\hat{b}^*\right)\hat{\alpha}_0\Gamma(1+\hat{b}^*/\hat{b}^*_0) \]

Authors suggested to generate a desired number say \( N=10000 \) of independent copies of \( G_{\mu} \) by plugging in the simulated values of \( \hat{\alpha}^* \) and \( \hat{\beta}^* \) based on \( N \) independent random samples each of size \( n \) from Weibull (1,1) distribution for given \( \hat{\alpha}_0 \) and \( \hat{b}_0 \). Then \( \alpha/2^{th} \) and \((1-\alpha/2)^{th} \) percentiles of the resultant sample of \( G_{\mu} \) can be used as CI for \( \mu \).

**RESULTS AND DATA ANALYSIS**

The results are summarized below according to objectives of the study:

**Comparison:**

Since a theoretical comparison of the estimators is not possible, a simulation study was conducted to compare the performance of the interval estimators. The CIs are compared with respect to their simulated average coverage probabilities and average widths based on the following simulation procedure.

**Simulation procedure:**

For various fixed sample sizes \( n=10,25,45 \), level of significance \( \alpha=0.05 \), shape parameter \( b=0.5,1,2,3 \) for which skewness of the underlying distribution varies from high to moderate and scale parameter \( a=0.5,1,5 \); 5000 simulated samples are generated from Weibull \((a,b)\) distribution. For each of these 5000 samples, lower and upper confidence limits \( L_i \) and \( U_i \) \( i=1,2,....,5000 \) are obtained for the population mean \( \mu \) for each of the above four CIs. It is noted that for GVA method, the computation of the CI based on each of the single simulated 5000 samples needs a further set of \( N \) (here we have taken \( N=10000 \)) samples of the same size from Weibull (1,1) distribution to generate independent copies of \( G_{\mu} \) as described in section 2.4. The simulated average width of each CI is obtained by averaging the 5000 quantities \( U_i - L_i \) \( i=1,2,....,5000 \).

The simulated average coverage probability is the proportion of these 5000 intervals that covered the value of the mean. Simulated average coverage probabilities and average widths (enclosed in parentheses) for all four CIs described in previous section for above mentioned sample sizes are reported in Table 1, for unknown scale and shape parameters \( a, b \). To enable a visual display and comparison among the methods, box plots of the average coverage probabilities and average widths over the set \( A = \{(a,b): a=0.5,1,5, b=0.5,1,2,3\} \) for sample sizes \( n=10,25,45 \) are plotted in Fig. 1. When the shape parameter \( b \) is known, all CIs perform excellent with respect to coverage probabilities. The coverages are almost close to nominal level 0.95 uniformly for all a and b mentioned above. Since all methods behave almost uniformly, these results are not displayed here.
Results of the simulation study:

The following prominent facts are clearly visible from the Table 1 and Fig. 1:

- Coverage probabilities of each CI increased as sample size and shape parameter increased and are invariant of scale parameter.
- Average widths of each CI decreased as both sample size and shape parameter increased and increased as scale parameter increased.
- When the shape parameter $b$ is unknown, GVA based CI outperforms the other three CIs almost uniformly, with respect to coverage probabilities, and more prominently for small shape parameter. The coverage probabilities are close to the nominal size 0.95.
- The average widths of NMMLE based CI were markedly smaller than its good competitor GVA, for small shape parameter and small sample size.
- For large shape parameter where the Weibull distribution is nearly symmetric and for large sample size, the performance of NMMLE and GVA based CIs were almost equivalent with respect to both average coverage probability and average width.
- The computational procedure for NMMLE based CI is simple than that for GVA based CI.
- Average widths of Wald CIs were smaller, however the corresponding coverage probabilities were very poor and were below the nominal size 0.95.

Based on above observations, we recommend GVA based

Table 1: Percentage simulated coverage probabilities and average widths (enclosed in parentheses) of 95% CI for Weibull mean for various sample sizes using WALD-LOG, WALD-CH, NMMLE, and GVA methods

<table>
<thead>
<tr>
<th>a</th>
<th>Method</th>
<th>n=10</th>
<th>n=25</th>
<th>n=45</th>
</tr>
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<tr>
<td></td>
<td></td>
<td>0.5</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>b</td>
<td>b</td>
<td>b</td>
</tr>
<tr>
<td></td>
<td>WALD-LOG</td>
<td>87.98</td>
<td>89.40</td>
<td>90.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(4.58)</td>
<td>(0.61)</td>
<td>(0.27)</td>
</tr>
<tr>
<td></td>
<td>WALD-CH</td>
<td>78.94</td>
<td>87.56</td>
<td>89.82</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.88)</td>
<td>(0.58)</td>
<td>(0.26)</td>
</tr>
<tr>
<td></td>
<td>NMMLE</td>
<td>93.50</td>
<td>93.90</td>
<td>94.10</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(16.56)</td>
<td>(0.87)</td>
<td>(0.33)</td>
</tr>
<tr>
<td></td>
<td>GVA</td>
<td>93.67</td>
<td>94.76</td>
<td>94.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(59.10)</td>
<td>(1.24)</td>
<td>(0.36)</td>
</tr>
<tr>
<td></td>
<td>WALD-LOG</td>
<td>87.26</td>
<td>89.68</td>
<td>90.00</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(9.15)</td>
<td>(1.23)</td>
<td>(0.54)</td>
</tr>
<tr>
<td></td>
<td>WALD-CH</td>
<td>79.10</td>
<td>86.48</td>
<td>90.32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(5.73)</td>
<td>(1.15)</td>
<td>(0.53)</td>
</tr>
<tr>
<td></td>
<td>NMMLE</td>
<td>93.56</td>
<td>93.82</td>
<td>93.70</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(38.50)</td>
<td>(1.74)</td>
<td>(0.66)</td>
</tr>
<tr>
<td></td>
<td>GVA</td>
<td>94.80</td>
<td>94.82</td>
<td>95.13</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(151.54)</td>
<td>(2.41)</td>
<td>(0.71)</td>
</tr>
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<td></td>
<td>WALD-LOG</td>
<td>87.44</td>
<td>89.74</td>
<td>89.98</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(44.24)</td>
<td>(6.20)</td>
<td>(2.66)</td>
</tr>
<tr>
<td></td>
<td>WALD-CH</td>
<td>79.90</td>
<td>86.74</td>
<td>89.50</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(29.11)</td>
<td>(5.84)</td>
<td>(2.65)</td>
</tr>
<tr>
<td></td>
<td>NMMLE</td>
<td>93.64</td>
<td>93.88</td>
<td>93.68</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(212.54)</td>
<td>(8.57)</td>
<td>(3.27)</td>
</tr>
<tr>
<td></td>
<td>GVA</td>
<td>93.67</td>
<td>94.28</td>
<td>94.60</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(591.04)</td>
<td>(12.66)</td>
<td>(3.60)</td>
</tr>
</tbody>
</table>

*aResults are obtained for unknown scale and shape parameters a, b.
Fig. 1: Box plots of simulated average coverage and average width for the methods WALD-LOG, WALD-CH, NMMLE, and GVA for various sample sizes (n) over the range of \(a=0.5,1.5\) and \(b=0.5,1,2,3\).
CI for all sample sizes even for small shape parameters. For large shape parameters, NMMLE based CI also can be used in practice.

Illustrative example:
In this section we illustrate the methods discussed in section 2 with a real data set. The data used here are the results of an experiment extracted from Nelson (1972). In this experiment, 11 specimens of a particular type of electrical insulating fluid are subjected to a constant voltage stress of 30 kilovats. The data set representing the failure time of each specimen is: 17.05, 22.66, 21.02, 175.88, 139.07, 144.12, 20.46, 43.40, 194.90, 47.30, 7.74. The Kolmogorov-Smirnov test to above data set for fitting Weibull distribution resulted p-value(2-tail) 0.61 indicating that Weibull is a good model for above data set. Here \( \hat{a} = 77.58 \), \( \hat{b} = 1.06 \), \( \tilde{b} = 0.9361 \), sample mean is 75.78 and sample standard deviation is 71.91. The MLE of shape parameter, \( \hat{b} \), indicates that the above data set is fairly skewed. The 95 per cent CIs for the mean time to failure (MTTF) and the corresponding CI widths (enclosed in parentheses) using WALD-LOG, WALD-CH, NMMLE, and GVA methods are, respectively [43.39,132.62] (89.23), [33.48,118.23] (84.75), [43.93, 159.90] (115.97) and [45.11, 198.59] (153.48). It seems that WALD-CH based CI has the narrowest width and the GVA based CI has the widest width.

Overall conclusion:
The generalized variable approach based CI exhibits markedly well performance even for small sample sizes for almost all parameter combinations considered in the simulation study for interval estimation of Weibull mean. For large shape parameters and large sample sizes, NMMLE and GVA based CIs perform almost similar. The Wald based CIs are not recommended due to their poor coverage probabilities.

**LITERATURE CITED**


